Relative Generalized Hamming Weight of q-ary Reed Muller codes

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Hamming weight

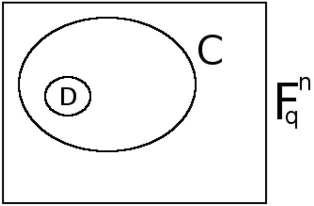
Let $\vec{c} \in \mathbb{F}_q^n$, $w_H(\vec{c}) = |\{i \in \{1, \ldots, n\}| : c_i \neq 0\}$.
For example: $w_H((1, 1, 0, 1, 2)) = 4$.
$d(C) = \min_{\vec{c} \in C \setminus \{\vec{0}\}} w_H(\vec{c})$

Generalized Hamming weight

Let $D \subseteq \mathbb{F}_q^n$ be a vector space.
$\mathrm{supp}(D) = \{i \in \{1, \ldots, n\} :$ there exists $\vec{c} \in D$ s.t. $c_i \neq 0\}$.
Let $D = \mathrm{span}_{\mathbb{F}_2}\{(1,1,0,0,0),(0,1,0,0,1)\}$, then
$\mathrm{supp}(D) = \{1,2,5\}$.

Let C be a linear code over $\mathbb{F}_q^n$. For $t = 1, \ldots, \dim(C)$, the t-th generalized Hamming weight of C is

$$d_t(C) = \min\{|\mathrm{supp}(D)| : D \text{ subspace of } C \text{ with } \dim(D) = t\}.$$



Generalized Hamming weight

Let $D \subseteq \mathbb{F}_q^n$ be a vector space.
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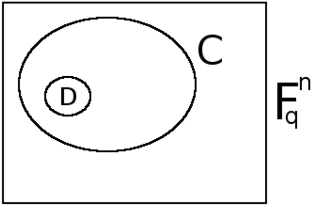
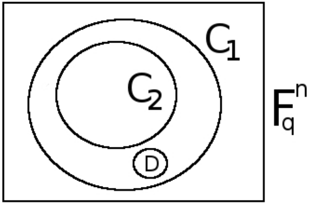


Relative generalized hamming weight

Let $C_2 \subsetneq C_1$ be linear codes over $\mathbb{F}_q^n$. For $m = 1, \ldots, \dim(C_1) - \dim(C_2)$, the m-th relative generalized Hamming weight of C_1 with respect to C_2 is

$$M_m(C_1, C_2) = \min\{|\mathrm{supp}(D)| : D \text{ subspace of } C_1,$$

$$C_2 \cap D = \{\vec{0}\}, \dim(D) = m\}.$$



q-ary Reed-Muller codes

Definition

Let u be a non-negative integer, and s a positive integer. Write $\{P_1, \ldots, P_{n=q^s}\} = (\mathbb{F}_q)^s$. The q-ary reed-Muller code of order u in s variables is defined by

$$RM_q(u, s) = \{(f(P_1), \ldots, f(P_n)) : f \in \mathbb{F}_q[X_1, \ldots, X_s],$$

$$\deg(f) \leq u, \deg_{X_i}(f) < q \text{ for all } i\}.$$

$\mathbb{F}_5[X, Y]$

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

$RM_5(5, 2)$

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

$RM_5(5,2)$ and $RM_5(3,2)$

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

Where is D?

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

Footprint

Definition

Let $J \subseteq \mathbb{F}_q[X_1, \ldots, X_s]$ be an ideal and $\prec$ be a fixed monomial ordering. Denote by $\mathcal{M}(X_1, \ldots, X_s)$ the monomials in the variables $X_1, \ldots, X_s$. The footprint of J with respect to $\prec$ is the set

$$\Delta_{\prec}(J) = \{M \in \mathcal{M}(X_1, \ldots, X_s) \mid M \text{ is not the leading monomial of any polynomial in } J\}.$$

Footprint

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

$$\Delta_{\prec}\left(\left\langle XY^3, X^5, Y^5\right\rangle\right)$$

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

$$\Delta_{\prec}\left(\left\langle XY^3, X^3Y, X^5, Y^5\right\rangle\right)$$

Footprint

Lemma

Let $F_1, \ldots, F_m \in \mathbb{F}_q[X_1, \ldots, X_s]$. The number of common zeros of $F_1, \ldots, F_m$ over $\mathbb{F}_q$ is $\left|\Delta_{\prec}(\langle F_1, \ldots, F_s, X_1^q - X_1, \ldots, X_m^q - X_m \rangle)\right|$ (here $\prec$ is any monomial ordering).

Footprint

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

$$|\Delta_{\prec}\left(\left\langle XY^3, X^5, Y^5\right\rangle\right)| = 17$$

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

$$|\Delta_{\prec}\left(\left\langle XY^3, X^3Y, X^5, Y^5\right\rangle\right)| = 13$$

Partial order

The partial order $\preceq_{\mathrm{P}}$ on the monomials is defined by:

$$X^{\vec{a}} \preceq_{\mathrm{P}} X^{\vec{b}} \iff a_i \leq b_i \text{ for all } i \in \{1, \ldots, s\}.$$

The upward shadow of $X^{\vec{a}}$ is

$$\nabla X^{\vec{a}} = \{X^{\vec{b}} \mid X^{\vec{b}} \succeq_{\mathrm{P}} X^{\vec{a}}, \deg_{X_i}(X^{\vec{b}}) < q\}.$$

Let A be a set of monomials, we define $\nabla A = \bigcup_{X^{\vec{a}} \in A} \nabla X^{\vec{a}}$.

Upward shadow

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

∇XY^3

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

$\nabla\{XY^3, X^3Y\}$

Footprint bound

Corollary

Let D be a subspace of $(\mathbb{F}_q)^n$ of dimension m generated by the evaluation of the polynomials $F_1, \ldots, F_m$. W.l.o.g. we assume $\mathrm{lm}(F_t) = X^{\vec{a}_t}$ for any t and $\vec{a}_t \neq \vec{a}_j$ for $t \neq j$. We have that:

$$|\mathrm{supp}(D)| \geq \left| \nabla \{X^{\vec{a}_1}, \ldots, X^{\vec{a}_m}\} \right|.$$

Footprint

$$F_1(X,Y) = XY^3 + X + Y, F_2(X,Y) = X^3Y + 3X^2$$

$$D_1 = \mathrm{span}_{\mathbb{F}_q}\{\varphi(F_1)\}, D_2 = \mathrm{span}_{\mathbb{F}_q}\{\varphi(F_1), \varphi(F_2)\}$$

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

$$|\mathrm{supp}(D_1)| \geq \left|\nabla XY^3\right| = 8$$

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

$$|\mathrm{supp}(D_2)| \geq \left|\nabla\{XY^3, X^3Y\}\right| = 12$$

Footprint bound is sometime sharp

For any choice of distinct $l_1, \ldots, l_m$ there exists some subspaces D for which this bound is sharp.

Proposition

Let $1 \leq l_1 < l_2 < \ldots < l_m \leq n$ *be integers. Then*

$$\min\{|\mathrm{supp}(D)| \mid D = \mathrm{span}_{\mathbb{F}_q}\{\varphi(F_1), \ldots, \varphi(F_m)\} \textit{ with}$$

$$\mathrm{lm}(F_i) = X^{\vec{a}_{l_i}} \textit{ for } i = 1, \ldots, m\} = \left|\nabla\{X^{\vec{a}_{l_1}}, \ldots, X^{\vec{a}_{l_m}}\}\right|.$$

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

$$M_m(RM_q(u_1,s), RM_q(u_2,s)) = \min\{|\mathrm{supp}(D)| : D \text{ subspace of } RM_q(u_1,s), RM_q(u_2,s) \cap RM_q(u_1,s) = \{\vec{0}\}, \dim(D) = m\}$$

$$= \min\{|\nabla A| : A \subseteq F(u_2+1, u_1), |A| = m\}$$

where $F(u_2+1, u_1) = \{X^{\vec{a}} : u_2 + 1 \leq \deg(X^{\vec{a}}) \leq u_1\}$

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

$$M_m(RM_q(u_1,s),RM_q(u_2,s)) = \min\{|\mathrm{supp}(D)| : D \text{ subspace of } RM_q(u_1,s), RM_q(u_2,s) \cap RM_q(u_1,s) = \{\vec{0}\}, \dim(D) = m\}$$

$$= \min\{|\nabla A| : A \subseteq F(u_2+1,u_1), |A| = m\}$$

where $F(u_2+1,u_1) = \{X^{\vec{a}} : u_2+1 \leq \deg(X^{\vec{a}}) \leq u_1\}$

First RGHW

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

Second RGHW

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

Third RGHW

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

Y^4	Y^4	XY^4			
Y^3		XY^3	X^2Y^3		
Y^2			X^2Y^2	X^2Y^2	
Y				X^3Y	X^3Y
1					X^4
	1	X	X^2	X^3	X^4

$(\mathbb{F}_5)^2$

Y^4	Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
Y^3	Y^3	XY^3	X^2Y^3	X^3Y^3	X^4Y^3
Y^2	Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	Y	XY	X^2Y	X^3Y	X^4Y
1	1	X	X^2	X^3	X^4
	1	X	X^2	X^3	X^4

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

$RM_5(5,2)$

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3	5	4	3		
Y^2	9	8	7	6	
Y	14	13	12	11	10
1	19	18	17	16	15
	1	X	X^2	X^3	X^4

$F(0,8)$, $F(0,5)$, $F(4,5)$

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3	5	4	3		
Y^2	9	8	7	6	
Y	14	13	12	11	10
1	19	18	17	16	15
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3		4	3		
Y^2			6	5	
Y				8	7
1					9
	1	X	X^2	X^3	X^4

$F(0,8)$, $F(0,5)$, $F(4,5)$

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3	5	4	3		
Y^2	9	8	7	6	
Y	14	13	12	11	10
1	19	18	17	16	15
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3		4	3		
Y^2			6	5	
Y				8	7
1					9
	1	X	X^2	X^3	X^4

First RGHW

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3	5	4	3		
Y^2	9	8	7	6	
Y	14	13	12	11	10
1	19	18	17	16	15
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3		4	3		
Y^2			6	5	
Y				8	7
1					9
	1	X	X^2	X^3	X^4

$$M_1(C_1, C_2) = 4 - 1 + 1 = 4$$

Second RGHW

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3	5	4	3		
Y^2	9	8	7	6	
Y	14	13	12	11	10
1	19	18	17	16	15
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3		4	3		
Y^2			6	5	
Y				8	7
1					9
	1	X	X^2	X^3	X^4

$$M_2(C_1, C_2) = 5 - 2 + 2 = 5$$

Third RGHW

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3	5	4	3		
Y^2	9	8	7	6	
Y	14	13	12	11	10
1	19	18	17	16	15
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3		4	3		
Y^2			6	5	
Y				8	7
1					9
	1	X	X^2	X^3	X^4

$$M_3(C_1, C_2) = 8 - 3 + 3 = 8$$

Fourth RGHW

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3	5	4	3		
Y^2	9	8	7	6	
Y	14	13	12	11	10
1	19	18	17	16	15
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3		4	3		
Y^2			6	5	
Y				8	7
1					9
	1	X	X^2	X^3	X^4

$$M_4(C_1, C_2) = 9 - 4 + 4 = 9$$

Fifth RGHW

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3	5	4	3		
Y^2	9	8	7	6	
Y	14	13	12	11	10
1	19	18	17	16	15
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3		4	3		
Y^2			6	5	
Y				8	7
1					9
	1	X	X^2	X^3	X^4

$$M_5(C_1, C_2) = 12 - 6 + 5 = 11$$

Sixth RGHW

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3	5	4	3		
Y^2	9	8	7	6	
Y	14	13	12	11	10
1	19	18	17	16	15
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3		4	3		
Y^2			6	5	
Y				8	7
1					9
	1	X	X^2	X^3	X^4

$$M_6(C_1, C_2) = 13 - 7 + 6 = 12$$

Seventh RGHW

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3	5	4	3		
Y^2	9	8	7	6	
Y	14	13	12	11	10
1	19	18	17	16	15
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3		4	3		
Y^2			6	5	
Y				8	7
1					9
	1	X	X^2	X^3	X^4

$$M_7(C_1, C_2) = 16 - 10 + 7 = 13$$

Eigth RGHW

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3	5	4	3		
Y^2	9	8	7	6	
Y	14	13	12	11	10
1	19	18	17	16	15
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3		4	3		
Y^2			6	5	
Y				8	7
1					9
	1	X	X^2	X^3	X^4

$$M_8(C_1, C_2) = 17 - 11 + 8 = 14$$

Ninth RGHW

Y^4	5	4	3	2	1
Y^3	10	9	8	7	6
Y^2	15	14	13	12	11
Y	20	19	18	17	16
1	25	24	23	22	21
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3	5	4	3		
Y^2	9	8	7	6	
Y	14	13	12	11	10
1	19	18	17	16	15
	1	X	X^2	X^3	X^4

Y^4	2	1			
Y^3		4	3		
Y^2			6	5	
Y				8	7
1					9
	1	X	X^2	X^3	X^4

$$M_9(C_1, C_2) = 21 - 15 + 9 = 15$$

RGHWs of q-ary Reed-Muller codes

$$F(a,b) = \{(a_1,\ldots,a_s) \in \{0,\ldots,q-1\}^s : a \leq \sum_{i=1}^{s} a_i \leq b\}$$

Theorem

Let $C_2 = RM_q(u_2,s) \subsetneq C_1 = RM_q(u_1,s)$. Let $\vec{a}$ be the m-th element in $F(u_2+1,u_1)$ using the anti lexicographic order. Because $F(u_2+1,u_1) \subseteq F(0,u_1) \subseteq F(0,s(q-1))$, then there exist r and t such that $\vec{a}$ is the r-th element in $F(0,u_1)$ and the t-th element in $F(0,s(q-1))$ using the anti lexicographic order. Then

$$M_m(C_1,C_2) = t - r + m.$$

GHWs of q-ary Reed-Muller codes

Theorem

Let $C_1 = RM_q(u_1, s)$. Let $\vec{a}$ be the r-th element in $F(0, u_1)$ using the anti lexicographic order. Because $F(0, u_1) \subseteq F(0, s(q-1))$, then there exist t such that $\vec{a}$ is the t-th element in $F(0, s(q-1))$ using the anti lexicographic order. Then

$$d_r(C_1) = t.$$

Comparison between GHW and RGHW

Let $C_1 = RM_5(5,2)$ and $C_2 = RM_5(3,2)$ then we have:

r/m	$d_r(C_1)$	$M_m(C_1, C_2)$
1	4	4
2	5	5
3	8	8
4	9	9
5	10	11
6	12	12
7	13	13
8	14	14
9	15	15
10	16	
11	17	
12	18	
13	19	
14	20	
15	21	
16	22	
17	23	
18	24	
19	25	

Comparison between GHW and RGHW

Let $C_1 = RM_5(4, 2)$ and $C_2 = RM_5(3, 2)$ then we have:

r/m	$d_r(C_1)$	$M_m(C_1, C_2)$
1	5	5
2	9	9
3	10	12
4	13	14
5	14	15
6	15	
7	17	
8	18	
9	19	
10	20	
11	21	
12	22	
13	23	
14	24	
15	25	

Thank you for your attention!